Matrix Representation of Graph Model for Conflict Resolution on Maritime Threat Response System of Systems

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Abstract. This paper demonstrates a matrix representation of multi-decision-maker (DM) Graph Model for Conflict Resolution (GMCR), which can identify equilibrium by stability analysis against each solution concept. More specifically, in addition to preference, joint unilateral movement and improvement matrices, four stability definitions are depicted in explicit matrix format to remedy the coding weakness of logical methodology. The complete modeling procedure is given to develop extensive algorithm adapt to complex game playing scenario. In this context, the Maritime Threat Response Project is a protection from surface-based terrorist threats, each function part of which served as decision maker and different option combination served as states. As a result, solution concepts are defined within matrix format and equilibrium is found by four kinds of stability analysis.

Introduction

Graph model for conflict resolution (GMCR) is a simple yet powerful methodology to analyze strategic conflicts, in which no less than two decision makers (DMs) hold different preferences and interact with each other [1][1]. It is pointed in the books edited by Hipel [2] that GMCR not only provides an adaptable framework for defining, comparing and characterizing solution concepts but also handles irreversible moves and models easily in the real world. As pointed out by Kilgour and Hipel [3], only relative preferences are required to calibrate a graph model, which can accommodate any finite number of decision makers (DMs) and states [4].Within the decision support system GMCR II [5],[6], anyone can create his own conflict model. However, the solution concepts in the graph model have been defined logically, which is hard to code for new models or solution concepts.

In this paper, we discuss four solution concepts named Nash stability, general metarationality (GMR), symmetric metarationality (SMR), and sequential stability (SEQ), all of which are defined in matrix representation. First, the basic matrix representation of GMCR is defined in section II, followed by the stability definition in matrix format and the complete algebraic analysis procedure in section III. Then, MTR SoS is employed in section IV using matrix representation method and the result reveals the best solution is stable for all DMs. Finally, some conclusions and future research insights are provided in section V.

Matrix Modeling of Graph Model For Conflict Resolution

Multi-decision-maker Graph Model GMCR methodology provides an efficient and flexible extension of metagame analysis, which requires minimal information in terms of decision makers as well as possible movements among feasible states [7][7]. Briefly speaking, to develop a graph model for conflict resolution, the essential components are as follow:

1) A finite set of DMs N, whose element number is n;
2) A finite set of states S, whose element number is m;
3) The preference P_i of each DM $i (i \in N)$ about states S;
4) A digraph $G_i = \{S, A_i\}$, where A_i means the set of oriented arcs of DM i;

Here, simple preference description is considered that the above pair of relations are strongly complete. Then, matrix representation of the multi-decision-maker graph model is depicted by the follow definitions and formulations.

J_i : a UM (unilateral movement) matrix, which is written as $m \times m$ 0-1 matrix.

$R_i(s)$: DM i 's reachable list from state s within one step move, which is written as a 0-1 row vector.

J_i^+ : a UI (unilateral improvement) matrix, which is written as $m \times m$ 0-1 matrix.

$R_i^+(\mathrm{s})$: DM i 's improvement list from state s within one step move, which is written as a 0-1 row vector.

e_s^{T} : the transpose of the sth standard basis vector of the m-dimentional Euclidean space.

$$J_i(s,q) = \begin{cases} 1, \text{ if } q \in R_i(\mathrm{s}) \\ 0, \text{ otherwise} \end{cases}, R_i(\mathrm{s}) = e_s^{\mathrm{T}} \cdot J_i \quad \text{......(1)}$$

$$J_i^+(s,q) = \begin{cases} 1, \text{ if } q \in R_i(\mathrm{s}) \text{ and } q \succ_i s \\ 0, \text{ otherwise} \end{cases}, R_i^+(\mathrm{s}) = e_s^{\mathrm{T}} \cdot J_i^+ \quad \text{......(2)}$$

Preference Rrelation Matrix Preference plays a crucial role in decision analysis, which needs to define a series of matrices to depict the movements in one step by a DM among a set of states. In this paper, some basic calculation definitions should be clarified firstly.

1) Hadamard product: both M and G are $m \times m$ matrices, and W is the Hadamard product of them.

$$W = M \circ G, \text{ i.e. } W(s,q) = M(s,q) \cdot G(s,q) \quad \text{......(3)}$$

2) Union function: both M and G are $m \times m$ matrices, H is the union of them.

$$H = M \vee G, \text{ i.e. } H(s,q) = \begin{cases} 1, \text{ if } M(s,q) + G(s,q) \neq 0 \\ 0, \text{ otherwise} \end{cases} \quad \text{......(4)}$$

3) Sign function:

$$sign[M(s,q)] = \begin{cases} 1, & M(s,q) > 0 \\ 0, & M(s,q) = 0 \\ -1, & M(s,q) < 0 \end{cases} \quad \text{......(5)}$$

Therefore, preference relation matrices for DM i are defined as follow.

$$P_i^+(s,q) = \begin{cases} 1, \text{ if } q \succ_i s \\ 0, \text{ otherwise} \end{cases}, \quad P_i^-(s,q) = \begin{cases} 1, \text{ if } s \succ_i \mathrm{q} \\ 0, \text{ otherwise} \end{cases}, \quad P_i^=(s,q) = \begin{cases} 1, \text{ if } s \sim_i q \\ 0, \text{ otherwise} \end{cases} \quad \text{......(6)}$$

So the relationship among the unilateral improvement matrices J_i^+, the unilateral move matrices J_i and preference relation matrix can be demonstrated as follow.

$$J_i^+ = J_i \circ P_i^+ \quad \text{......(7)}$$

Matrix Representation of Stability Solutions

GMCR method pursues to reach possible stable resolutions that no one intends to move from the status quo. Stability analysis is a systematic examination of permissible moves and countermoves by all DMs, which is also a precise mathematical calculation whether a DM would prefer to stay at a state or move away from it.

Joint Movement and Improvement Matrices First of all, there is a hypothesis that in a legal sequence of UMs for a group, any DM may move more than once, but not twice consecutively. Then, to verify the following calculation procedure, some definitions need be explicit. $\Omega_H(s,s_1)$, $\Omega_H^+(s,s_1)$ means the set of all last DMs in legal UM or UI sequence from s to s_1, respectively. $R_H(s)$, $R_H^+(s)$

denotes the set of states can be reached by any legal sequence of UMs or UIs, starting at state s. Both of them are defined by inductive formulations.

$R_H = \cup_{s\in S} R_H(s)$ as joint UMs, $R_H^+ = \cup_{s\in S} R_H^+(s)$ as joint UIs ………………………………(8)

In this paper, we try to find matrix representation about $R_H(s)$ and $R_H^+(s)$, before which two $m\times m$ matrices are defined as follow.

$$M_i^{(t)}(s,q) = \begin{cases} 1, & \text{if } q\in S \text{ is reachable from } s\in S \text{ in exactly} \\ & t \text{ legal UMs with first mover DM } i \\ 0, & \text{otherwise} \end{cases}$$
$$M_i^{(t,+)}(s,q) = \begin{cases} 1, & \text{if } q\in S \text{ is reachable from } s\in S \text{ in exactly} \\ & t \text{ legal UIs with first mover DM } i \\ 0, & \text{otherwise} \end{cases} \quad \text{..(9)}$$

The two $m\times m$ matrices can be depicted by inductive formulations as follow, which is proved by Hai *et al*. in [8].

$$M_i^{(t)} = sign(J_i \cdot (\underset{j\in H-\{i\}}{\vee} M_j^{(t-1)})), \text{while } M_i^{(1)} = J_i, \quad M_i^{(t,+)} = sign(J_i^+ \cdot (\underset{j\in H-\{i\}}{\vee} M_j^{(t-1,+)})), \text{while } M_i^{(1,+)} = J_i^+$$
…………………………………………………………………………………………..(10)

To represent possible movements in the graph model with n DMs, two $m\times m$ matrices M_H and M_H^+ are shown as follow, where R_H and R_H^+ are 0-1 vectors.

$$M_H(s,q) = \begin{cases} 1, & \text{if } q\in R_H(s) \\ 0, & \text{otherwise} \end{cases} \qquad M_H^+(s,q) = \begin{cases} 1, & \text{if } q\in R_H^+(s) \\ 0, & \text{otherwise} \end{cases} \quad \text{……………………….…….(11)}$$

$R_H(s) = e_s^{\mathrm{T}} \cdot M_H$, $R_H^+(s) = e_s^{\mathrm{T}} \cdot M_H^+$ ……………………………………………………........(12)

The joint movement matrix M_H and the joint improvement matrix M_H^+ can be expressed as follow, while δ, δ_1 denotes the number of iterations required to find $R_H(s)$ and $R_H^+(s)$, so $\delta \le L$ and $\delta_1 \le L_1$, while L, L_1 means the sum of UM and UI arcs in all graphs, respectively.

$$M_H = \overset{\delta}{\underset{t=1}{\vee}} \underset{i\in H}{\vee} M_i^{(t)}, \quad M_H^+ = \overset{\delta_1}{\underset{t=1}{\vee}} \underset{i\in H}{\vee} M_i^{(t,+)} \quad \text{…………………………………………………………(13)}$$

Matrix Representation of Stability Solutions General definitions of stability include 4 different kinds, named Nash stability, general metarational stability (GMR), symmetric metarational stability (SMR), and sequential stability (SEQ). The logical definitions as follow are concluded from [11], and the relationship among them is shown in Fig. 1.

1) Nash stability: s is Nash stable for DM i iff R_i^+(s)=∅
2) GMR stability: $\forall s_1 \in R_i^+(s) \Rightarrow \exists s_2 \in R_{N-\{i\}}(s_1)$, $s \succ_i s_2$
3) SMR stability: $\forall s_1 \in R_i^+(s) \Rightarrow \exists s_2 \in R_{N-\{i\}}(s_1)$, $s \succ_i s_2$, $s \succ_i s_3$, $s_3 \in R_i(s_2)$
4) SEQ stability: $\forall s_1 \in R_i^+(s) \Rightarrow \exists s_2 \in R_{N-\{i\}}^+(s_1)$, $s \succ_i s_2$

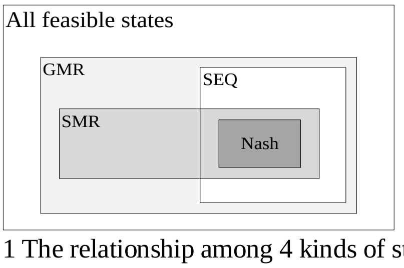

Fig. 1 The relationship among 4 kinds of stability.

The Programming Procedure When it comes to coding, one inevitable problem is the computational complexity. Xu *et al*. [9] calculate the computational complexity using GMR in 2-DM models as an example, which is $O(m^2)$, where m is the number of states. In terms of n-DMs decision analysis, the computational complexity is $L\cdot(n-1)\cdot O(m^3)$, which is acceptable as polynomial-time effective algorithm.

The procedure to solve this kind of problem is as follow and the flow chart is given in Fig. 2.

Step1: verify the set of DM N , the feasible states (by each DMs options portfolio), and the preference sort;

Step2: construct UM, UI matrices of each DM J_i and J_i^+;

Step3: construct preference matrices of each DM P_i^+ and $P_i^{-,=}$ according to the preference sort.

Step4: calculate M_H and M_H^+ , which is an iteration process using inductive formulations.

Step5: analyze 4 kinds of stabilities for each DM and find the equilibrium.

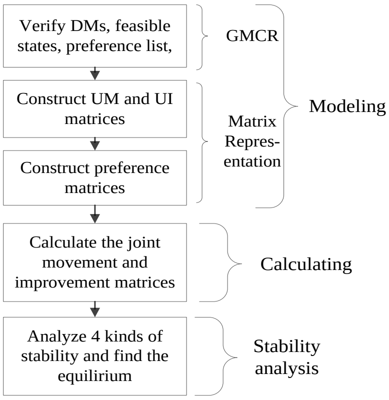

Fig. 2 The flow chart of problem solving procedure.

Maritime Threat Response Systems of Systems

In this section, one illustrative example is provided by the method of matrix representation, which is mainly based on the national maritime threat response (MTR) system of systems (SoS) by Huynh et al. [10],[11]. Three main mission domains are included: Weapons of Mass Destruction (WMD), Ship used As a Weapon (SAW), Small Boat Attach against high value unit (SBA). n the graph model, they are different decision makers: C4ISR capability, Prepare Battle Space (PBS) and Detect capability, Engage capability [12].

Hence, $N=\{DM_1,DM_2,DM_3\}, n=3$. Each DM represents an area of capability in consist of several options, $O=\{O_1,O_2,...,O_9\}$. The details are show in Fig. 3. The possible state is a feasible combination of Y and N, the universal set of which is $2^9=512$. However, just few of them are feasible. DM_1: YNYN, NYYN, YNNY, NYNY; DM_2: YN, NY; DM_3: YNN, YYN, YNY, YYY.

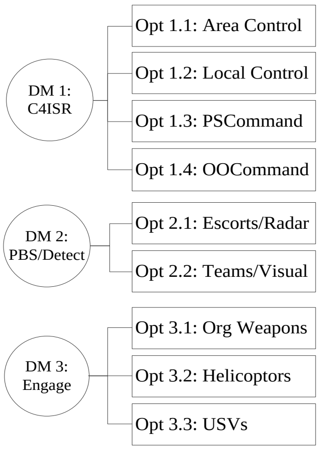

Fig. 3 The DMs' Option Combination.

Therefore, we have 32($4\times2\times4=32$) feasible states shown in Table 1,. In this example, each DM can change the state by choosing any feasible combination of his own without other limitations, so the graph model is obtained in Fig. 4.

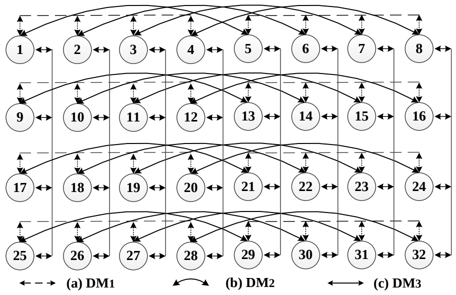

Fig. 4 The Graph Model for Movements of MTR.

Table 1 Feasible States List

DM	Option	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
DM1	Option 1.1	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N	N
	Option 1.2	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y
	Option 1.3	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
	Option 1.4	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
DM2	Option 2.1	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N
	Option 2.2	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y
DM3	Option 3.1	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
	Option 3.2	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y
	Option 3.3	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y

Table 2 The Relative Preference Ranking for Each DM

DM	Preference Ranking
C4ISR	$27 \succ 11 \succ 31 \succ 15 \succ 28 \succ 12 \succ 32 \succ 16 \succ 19 \succ 3 \succ 23 \succ 7 \succ 20 \succ 4 \succ 24 \succ 8 \succ 29 \succ 13 \succ 25 \succ 9 \succ 30 \succ 14 \succ 26 \succ 10 \succ 5 \succ 21 \succ 1 \succ 17 \succ 6 \succ 22 \succ 2 \succ 18$
PBS	$28 \succ 27 \succ 12 \succ 11 \succ 26 \succ 25 \succ 10 \succ 9 \succ 4 \succ 3 \succ 20 \succ 19 \succ 2 \succ 1 \succ 18 \succ 17 \succ 30 \succ 29 \succ 14 \succ 13 \succ 32 \succ 31 \succ 16 \succ 15 \succ 22 \succ 21 \succ 6 \succ 5 \succ 24 \succ 23 \succ 8 \succ 7$
Engage	$28 \succ 27 \succ 26 \succ 25 \succ 12 \succ 11 \succ 10 \succ 9 \succ 32 \succ 31 \succ 30 \succ 29 \succ 16 \succ 15 \succ 14 \succ 13 \succ 4 \succ 3 \succ 2 \succ 1 \succ 20 \succ 19 \succ 18 \succ 17 \succ 6 \succ 5 \succ 8 \succ 7 \succ 24 \succ 23 \succ 22 \succ 21$

In this case study, the final preference rankings are obtained as shown in Table 2. According to which, the construction of matrices J_i, J_i^+, P_i^+, $P_i^{-,=}$, for $i = 1,2,3$, can easily get by the information in figure4. Let $N = \{1,2,3\}$ and $H = N - \{i\}$, the joint movement matrices and the joint improvement matrices are obtained by inductive formulations as follows, where $M_i^{(1)} = J_i$, $M_i^{(1,+)} = J_i^+$, and $\delta = L = 128$, L is the sum of arcs in the digraph (Fig. 4). Consequently, from the result table, state 27 is the best choice because it is equilibria for all four solution concepts.

Table 3 Stability Anadlysis Result of The MTR System of Systems

States	Nash				GMR				SMR				SEQ			
	DM1	DM2	DM3	E	DM1	DM2	DM3	E	DM1	DM2	DM3	E	DM1	DM2	DM3	E
1		√				√				√				√		
2		√				√				√				√		
3	√	√			√	√			√	√			√	√		
4		√				√				√				√		
5																
6																
7	√				√				√				√			
8																
9		√				√	√			√	√			√		
10		√				√	√			√	√			√		
11	√	√			√	√	√	√	√	√	√	√	√	√		
12		√			√	√	√	√	√	√	√	√		√		
13																
14																
15	√				√				√				√			
16					√				√							
17		√				√				√				√		
18		√				√				√				√		
19	√	√			√	√			√	√			√	√		
20		√				√				√				√		
21																
22																
23	√				√				√				√			
24																
25		√	√			√	√			√	√			√	√	
26		√	√			√	√			√	√			√	√	
27	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
28		√	√		√	√	√	√	√	√	√	√		√	√	
29			√				√				√				√	
30			√				√				√				√	
31	√		√		√		√		√		√		√		√	
32			√		√		√		√		√				√	

Conclusion and future work

The author puts forward the matrix representation of graph model for conflict resolution is put forward to handle the large-scale multistakeholder decision making, which requires strict mathematical deduction and standardized process. Such a novel, efficient yet flexible approach is designed for multistakeholder, and thereby conceptualizes the decision making process in a standard mathematical way of matrix representation. As demonstrated by the case of MTR SoS, it is of significance to employ a rigorous logical process from information collection, modeling to stability analysis, which is easy to modify for different solution concepts and various stability analysis for large-scale conflict resolution problem.

Furthermore, the explicit algebraic expressions for calculations of multistakeholder GMCR with uncertainty preference, fuzzy preference, or strength of preference should be taken into consider, as well as programming realization of coalition analysis or status quo analysis based on the algebraic characteristics.

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